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Uncountably many pairwise disjoint copies of one metrizable compactum in another

Eric K. van Douwen

Abstract

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We prove that if K is a compact metrizable space and if X is separable and completely metrizable and contains uncountably many pairwise disjoint homeomorphs of K then X contains a copy of ${}^{\omega}2 \times K$. We also present applications of this result.

Keywords: Cantor set, embedding, compact metrizable space.

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1. Results

Let ${}^{\omega}2$ denote the Cantor discontinuum. In this paper we prove the following.

Theorem 1. *Let X be separable and completely metrizable, and let K be compact. Then $K \times {}^{\omega}2$ embeds into X if (and trivially only if) X has an uncountable pairwise disjoint family of subspaces each homeomorphic to K.*

The motivation for proving this is that it has the following corollary.

Theorem 2. *Let X be a separable completely metrizable space which has a point p and a collection $\mathscr{A}$ of arcs ($\equiv$ copies of $I = [0, 1]$) such that $X = \bigcup \mathscr{A}$, and p is an endpoint of each member of $\mathscr{A}$, and $A \cap A' = \{p\}$ for every two distinct $A, A' \in \mathscr{A}$. Then the following are equivalent:*

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(1) *X is rational, i.e., the collection of open sets that have a countable boundary is a base*;

(2) *X is rational at p (self-explanatory)*;

(3) *$\mathcal{A}$ is countable (= at most countable)*;

(4) *X is the union of some countable collection of arcs*; *and*

(5) *$I \times {}^{\omega}2$ does not embed in X.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1) ⇒ (5) are easy, if not trivial, and (5) ⇒ (3) follows from Theorem 1. □

This answers Lelek's question of whether (1) and (5) are equivalent for compact one-dimensional X, [5, P15].

Theorem 1 partially generalizes the classical result that an uncountable continuous image of the irrationals[1] has a subspace homeomorphic to ${}^{\omega}2$, [3, p. 437]: let $|K| = 1$. (I have not investigated the question of whether one can weaken the condition in X in Theorem 1.) After proving Theorem 1 I discovered an analogous result by Roberts, that for nicely enough embedded[2] snake-like continua K in the plane Π there is a collection of 2^{ω} pairwise disjoint copies of K in Π, [7], especially since inspection of the proof reveals that $K \times {}^{\omega}2$ is shown to embed in Π. The proof of Theorem 1 is, like the proofs of the two results just mentioned, a Cantor tree type argument, which is made interesting by the fact that we have no information about how nicely the copies of K are embedded.[3] Also, they need not be equivalently embedded.

2. Conventions

We follow some common set theoretic short hand.

ω denotes the nonnegative integers, and n is $\{i \in \omega\colon i < n\}$ for $n \in \omega$. In particular, $0 = \emptyset$ and $2 = \{0, 1\}$.

If f is a function, then $\mathrm{dom}(f)$ denotes its domain, and $f^{\to}A$ denotes the image under f of a set A. Of course f equals its graph $\{\langle x, f(x)\rangle\colon x \in \mathrm{dom}(f)\}$, hence if g is another function then $g \supseteq f$ means that g extends f.

The set of all functions $A \to B$ is denoted by ${}^{A}B$, in particular ${}^{n}2$ denotes the set of all n-term sequences of 0's and 1's (so ${}^{0}2 = \{\emptyset\}$) and ${}^{\omega}2$ denotes the underlying

[1] Recall that every separable completely metrizable space is a continuous image of the irrationals, [3, p. 434].

[2] Bing has shown that every snake-like continuum can be embedded nicely enough, [2, Theorem 4, cf. Theorem 5].

[3] In this connection it is of interest to note the following corollary to Theorem 1: If P denotes the pseudo-arc, i.e., the unique hereditarily indecomposable snake-like metrizable continuum, [1, Theorem 1], then $P \times {}^{\omega}2$ embeds in P (proved earlier by Lelek, [4]), since P has a family of 2^{ω} pairwise disjoint nondegenerate subcontinua, being indecomposable, and they are all homeomorphic to P, [6], see also [1, Theorem 1].

set of the product of ω copies of 2. We use ${}^{<\omega}2$ to abbreviate $\bigcup_{n\in\omega}{}^{n}2$, the Cantor tree. Note that $\mathrm{dom}(s)=|s|$ for $s\in{}^{<\omega}2$, and that

$$\{\{x\in{}^{\omega}2\colon x\subseteq s\}\colon s\in{}^{<\omega}2\}$$

is the usual base for the product topology on ${}^{\omega}2$ if 2 is discrete. Finally, for $s\in{}^{<\omega}2$ and $i\in 2$ we use $s^\frown i$ to denote the concatenation of s and i, i.e., $s^\frown i=s\cup\{\langle |s|, i\rangle\}$.

Proof of Theorem 1. Let F be an uncountable set of embeddings of K into X such that $f^{\to}K\cap g^{\to}K=\emptyset$ for every two distinct $f, g\in F$.

Let $\langle\mathscr{A}_n\rangle_n$ be a sequence of finite open covers of K with $\mathscr{A}_0=\{K\}$ such that $\bigcup_n\mathscr{A}_n$ is a base for K, and such that $\mathscr{A}_{n+1}$ refines $\mathscr{A}_n$ for $n\in\omega$, and such that $\emptyset\notin\bigcup_n\mathscr{A}_n$. Then, since K is compact,

$$(\forall f\in F)(\forall\varepsilon>0)(\exists n\in\omega)(\forall A\in\textstyle\bigcup_{k\geq n}\mathscr{A}_k)[\mathrm{diam}(f^{\to}A)<\varepsilon] \qquad (*)$$

no matter which metric we use for X.

Let X carry a complete metric such that $\mathrm{diam}(X)<1$, let $\mathscr{B}$ be a countable base for X closed under finite unions and intersections. Note that

> for every uncountable pairwise disjoint collection $\mathscr{C}$ of compact subsets of X there are $B_0, B_1\in\mathscr{B}$ with $\bar{B}_0\cap\bar{B}_1=\emptyset$ such that $\{C\in\mathscr{C}\colon C\subseteq B_i\}$ is uncountable, for $i\in 2$. $\qquad(**)$

Indeed, if $\mathscr{S}=\{\langle C_0, C_1\rangle\in\mathscr{C}\times\mathscr{C}\colon C_0\neq C_1\}$ and $\mathscr{P}=\{\langle B_0, B_1\rangle\in\mathscr{B}\times\mathscr{B}\colon \bar{B}_0\cap\bar{B}_1=\emptyset\}$, then for each $\langle C_0, C_1\rangle\in\mathscr{S}$ there is $\langle B_0, B_1\rangle\in\mathscr{P}$ with $C_i\subseteq B_i$ for $i\in 2$, hence $(**)$ because $|\mathscr{C}|>\omega=|\mathscr{B}|$.

The following claim is a major part of our proof.

Claim. *There are a strictly increasing $\sigma:\omega\to\omega$ and a function $\Gamma:\bigcup_{n\in\omega}\mathscr{A}_{\sigma(n)}\times{}^{n}2\to\mathscr{B}$ and a function $\Phi:{}^{<\omega}2\to\mathscr{P}(F)$ such that*

(1) $\mathrm{diam}(\Gamma(A, s))<2^{-|s|}$ *for* $\langle A, s\rangle\in\mathrm{dom}(\Gamma)$;

(2) $\Gamma(A, s)\cap\Gamma(A', s')=\emptyset$ for $\langle A, s\rangle, \langle A', s'\rangle\in\mathscr{A}_{\sigma(n)}\times{}^{n}2$ *with* $s\restriction(n-1)=s'\restriction(n-1)$ *and with* $\bar{A}\cap\bar{A}'=\emptyset$ *or* $s\neq s'$;

(3) $\Phi(s)\subseteq\{f\in F\colon f^{\to}A\subseteq\Gamma(A, s)$ *for all* $A\in\mathscr{A}_{\sigma(|s|)}\}$;

(4) $\Phi(s)$ *is uncountable for all* $s\in{}^{<\omega}2$;

(5) $\Phi(s)\supseteq\Phi(s')$ *for all* $s, s'\in{}^{<\omega}2$ *with* $s\subseteq s'$.

Proof of Claim. We construct $\sigma(n)$, and $\Gamma\restriction\mathscr{A}_{\sigma(n)}\times{}^{<\omega}2$, and $\Phi\restriction{}^{n}2$ with induction on $n\in\omega$. Let $\sigma(0)=0$, and $\Gamma(K,\emptyset)=X$, and $\Phi(\emptyset)=F$. All conditions, as far as applicable, hold.

Now let $n>0$, and suppose $\sigma(k)$ and $\Gamma\restriction\mathscr{A}_{\sigma(k)}\times{}^{k}2$ and $\Phi\restriction{}^{k}2$ to be constructed for $k<n$.

We first determine $\sigma(n)$. Since ${}^{(n-1)}2$ is finite $(*)$ enables us to find $\sigma(n)>\sigma(n-1)$ such that

$$\Phi'(s)=\{f\in\Phi(s)\colon \mathrm{diam}(f^{\to}A)<2^{-n} \text{ for all } A\in\mathscr{A}_{\sigma(n)}\}$$

is uncountable for all $s\in{}^{(n-1)}2$.

We now construct auxiliary functions Γ_0, Φ'' and Γ_1, and also $\Phi \restriction {}^n2$.

Construction of Γ_0 and Φ''. For each $s \in {}^{(n-1)}2$ and $f \in \Phi'(s)$ the indexed collection $\langle f^{\to}\bar{A}: A \in \mathscr{A}_{\sigma(n)}\rangle$ is a finite collection of compact sets of diameter less than 2^{-n}, hence we can find $T_f : \mathscr{A}_{\sigma(n)} \to \mathscr{B}$ such that $f^{\to}A \subseteq T_f(A)$ and $\operatorname{diam}(T_f(A)) < 2^{-n}$, for $A \in \mathscr{A}_{\sigma(n)}$; and $T_f(A)^{-} \cap T_f(A')^{-} = \emptyset$ for $A, A' \in \mathscr{A}_{\sigma(n)}$ with $\bar{A} \cap \bar{A}' = \emptyset$.

(Note that if $C \subseteq X$ is compact then $\{B \in \mathscr{B}: B \subseteq C\}$ is a neighborhood base for C.)

Since there are only countably many functions $\mathscr{A}_{\sigma(n)} \to \mathscr{B}$ there is for each $s \in {}^{(n-1)}2$ a $T^s : \mathscr{A}_{\sigma(n)} \to \mathscr{B}$ such that

$$\Phi''(s) = \{f \in \Phi'(s): T_f = T^s\}$$

is uncountable. Define $\Gamma_0: \mathscr{A}_{\sigma(n)} \times {}^{(n-1)}2 \to \mathscr{B}$ by

$$\Gamma_0(A, s) = T^s(A) \quad \text{for } \langle A, s\rangle \in \mathscr{A}_{\sigma(n)} \times {}^{(n-1)}2.$$

Construction of Γ_1 and $\Phi \restriction {}^n2$. Because of (**) we can find for each $s \in {}^{(n-1)}2$ sets $B_{s^\frown 0}, B_{s^\frown 1} \in \mathscr{B}$ such that $\bar{B}_{s^\frown 0} \cap \bar{B}_{s^\frown 1} = \emptyset$, and

$$\Phi(s^\frown i) = \{f \in \Phi''(s): f^{\to}K \subseteq B_{s^\frown i}\}$$

is uncountable, for $i \in 2$. Define

$$\Gamma_1(A, s) = B_s \quad \text{for } \langle A, s\rangle \in \mathscr{A}_{\sigma(n)} \times {}^n2.$$

Having found Γ_0 and Γ_1 we define $\Gamma \restriction \mathscr{A}_{\sigma(n)} \times {}^n2$ by

$$\Gamma(A, s) = \Gamma_0(A, s \restriction n-1) \cap \Gamma_1(A, s) \quad \text{for } \langle A, s\rangle \in \mathscr{A}_{\sigma(n)} \times {}^n2.$$

Clearly all conditions, as far as applicable, hold.

This completes the construction of σ, Γ and ϕ.

We now are ready to construct an embedding $e: K \times {}^\omega2 \to X$. Without loss of generality $\sigma = \mathrm{id}_\omega$. First define $\mathscr{E}: K \times {}^\omega2 \to \mathscr{P}(\mathscr{P}(X))$ by

$$\mathscr{E}(y, t) = \{\Gamma(A, t \restriction n)^{-}: n \in \omega, \text{ and } A \in \mathscr{A}_n, \text{ and } y \in A\}.$$

We claim that

$$|\textstyle\bigcap \mathscr{E}(y, t)| = 1 \quad \text{for all } \langle y, t\rangle \in K \times {}^\omega2.$$

Indeed, fix $\langle y, t\rangle \in K \times {}^\omega2$. By (1) there is for each $n \in \omega$ an $E \in \mathscr{E}(y, t)$ with $\operatorname{diam}(E) < 2^{-n}$ since $\mathscr{A}_n$ covers K, hence it suffices to show that $\mathscr{E}(y, t)$ is centered: Given $A \in \mathscr{A}_n$ and $A' \in \mathscr{A}_{n'}$, with $y \in A \cap A'$ there are $n'' \geq \max\{n, n'\}$ and $A'' \in \mathscr{A}_{n''}$ with $y \in A'' \subseteq A \cap A'$. If $f \in \Phi(t \restriction n'')$ then also $f \in \Phi(t \restriction n)$ and $f \in \Phi(t \restriction n'')$, hence

$$\emptyset \neq f^{\to}A'' \subseteq f^{\to}A \cap f^{\to}A' \subseteq \Gamma(A, t \restriction n) \cap \Gamma(A', t \restriction n').$$

We therefore can define a map $e: K \times {}^\omega2 \to X$ by

$$\{e(y, t)\} = \textstyle\bigcap \mathscr{E}(y, t) \quad \text{for } y \in K,\ t \in {}^\omega2.$$

Since each $\mathscr{A}_n$ covers K it is clear from (1) that e is continuous. Also, e is an injection because of (2): Consider any two distinct $\langle y_0, t_0\rangle$, $\langle y_1, t_1\rangle \in K \times {}^{\omega}2$. There is an $n \geq 1$ such that $t_0 \restriction n = t_1 \restriction n$, and $t_0 \neq t_1$ or there are $A_0, A_1 \in \mathscr{A}_n$ with $\bar{A}_0 \cap \bar{A}_1 = \emptyset$, and $y_i \in A_i$ for $i \in 2$. Then $\Gamma(A_0, t_0 \restriction n)^- \cap \Gamma(A_1, t_1 \restriction n)^- = \emptyset$ by (2), hence $e(y_0, t_0) \neq e(y_1, t_1)$. $\square$

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Notes by the editor

This paper was originally submitted to Houston Journal of Mathematics on June 2, 1981. It was refereed and accepted for publication but van Douwen never submitted a revised version of his manuscript. The editor of Houston Journal of Mathematics has authorized me to publish the paper here and he has provided me with a copy of the report of the referee. I have made those changes and improvements suggested by the referee that I thought were appropriate.

Theorem 1 of this paper can also be proved by an elementary function space type argument. This was observed independently by Pol and Becker, van Engelen and van Mill ("*Disjoint embeddings of compacta*", to appear in Mathematika). Also, Becker, van Engelen and van Mill observed the following. If Det(Π_1^1) holds then every coanalytic space containing uncountably many copies of a compact space K contains a copy of $K \times {}^{\omega}2$. Also, there is an example of a σ-compact space containing uncountably many pairwise disjoint copies of the circle S^1 but not $S^1 \times {}^{\omega}2$, granting the existence of an uncountable coanalytic space without perfect subsets. Even for σ-compact metrizable spaces the answer to the question formulated in Section 1 of this paper is independent of the usual axioms of set theory.